The Variational Iteration Method and the Variational Homotopy Perturbation Method for Solving the KdV-Burgers Equation and the Sharma-Tasso-Olver Equation

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In this article, two powerful analytical methods called the variational iteration method (VIM) and the variational homotopy perturbation method (VHPM) are introduced to obtain the exact and the numerical solutions of the (2+1)-dimensional Korteweg-de Vries-Burgers (KdVB) equation and the (1+1)-dimensional Sharma-Tasso-Olver equation. The main objective of the present article is to propose alternative methods of solutions, which avoid linearization and physical unrealistic assumptions. The results show that these methods are very efficient, convenient and can be applied to a large class of nonlinear problems.

Key words: Variational Iteration Method; Variational Homotopy Perturbation Method; (2+1)-Dimensional Korteweg-de Vries-Burgers Equation; (1+1)-Dimensional Sharma-Tasso-Olver Equation.

1. Introduction

It is well known [1 – 48] that the nonlinear phenomena are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, and chemical physics. Searching for exact and numerical solutions, travelling wave solutions of nonlinear equations in mathematical physics play an important role in soliton theory [7, 39]. Recently, many new approaches to the nonlinear equations were proposed [1 – 48], such as Bäcklund transformation [13, 33, 34], the inverse scattering transform method [7, 44], tanh method [12], extended tanh method [14, 41], Adomian pade approximation [29], variational iteration method [6, 16, 25, 32], modified variational iteration method [5], homotopy perturbation method [17 – 19, 22, 46, 47], various Lindstedt-Poincare methods [20, 21], Adomian decomposition method [14, 8, 9, 11, 15, 23, 26, 28], F-expansion method [37], exp-function method [10, 38], the sine-cosine method [40], the Jacobi elliptic function expansion method [42 – 44], the complex hyperbolic function method [45], and the $\left(\frac{G'}{G}\right)$-expansion method [48]. Among all of the analytical methods in open literature, the variational iteration method which is a modified general Lagrange's multiplier method [5, 6, 16, 25, 32] has been shown to solve effectively, easily, and accurately a large class of nonlinear problems. The main feature of the method is that the solution of a mathematical problem without linearalization assumption is used as initial approximation or trial function, then a more highly precise approximation at some special points can be obtained. This approximation converges rapidly to an accurate solution. The homotopy perturbation method [17 – 19, 22, 46, 47], proposed first by He, was further developed and improved in [17 – 19]. This method yields a very rapid convergence of the solution series in the most cases. The objective of this article is to use a variational iteration method (VIM) and its combination with the homotopy perturbation method (HPM), which is called the variational homotopy perturbation method (VHPM), to solve the (2+1)-dimensional Korteweg-de Vries-Burgers (KdVB) equation [30]

$$(u_t + uu_x - qu_{xx} + \mu u_{xxx})_x + ru_{yy} = 0, \tag{1}$$

and the (1+1)-dimensional Sharma-Tasso-Olver equation [10]

$$u_t + \alpha(u^3)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0, \tag{2}$$

where μ, q, r, and α are constants. Molabahrami et al. [30] have applied the homotopy perturbation method for solving equation (1) and Bekir and Bo [10] have obtained the exact solutions for equation (2) using the exp-function method. In this article, we obtain the numerical solutions of (1) and (2) by applying He's variational iteration method and the variational homotopy perturbation method. The results reveal that the proposed methods are very efficient and simple.

2. Variational Iteration Method

To illustrate the basic concept of this technique, we consider the following general differential equation:

$$Lu+Nu=g, \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and g is the forcing term. According to the variational iteration method [5, 16], we can construct a correct functional as follows:

$$u_{n+1}=u_n+\int_0^t \lambda(\tau)[Lu_n+N\tilde{u}_n-g]\mathrm{d}\tau, \quad n\geq 0, \tag{4}$$

where λ is a Lagrange multiplier [5, 6, 16, 25, 32] which can be identified optimally via variational iteration method. The subscripts n denote the n^{th} approximation, $\tilde{u}_n$ is considered as restricted variation, i. e. $\delta\tilde{u}=0$. (4) is called a correct functional. The solution of the linear problem can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principals of the variational iteration method and its applicability for various kinds of differential equations are given in [5, 6, 16, 25, 32]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n\geq 0$, of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0. Consequently, the solution is given by

$$u=\lim_{n\to\infty} u_n. \tag{5}$$

3. Homotopy Perturbation Method

To illustrate the homotopy perturbation method, we consider a general equation of the type

$$L(u)=0, \tag{6}$$

where L is any integral or differential operator. We define a convex homotopy $H(u,p)$ by

$$L(u,p)=(1-p)F(u)+pL(u), \tag{7}$$

where $F(u)$ is a functional operator with known solution v_0 which can be obtained easily. It is clear that, for

$$H(u,p)=0, \tag{8}$$

we have

$$H(u,0)=F(u), \quad H(u,1)=L(u). \tag{9}$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(f,1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u)=0$ continuously deforms the original problem $L(u)=0$. The embedding parameter $p\in[0,1]$ can be considered as an expanding parameter [17 – 19, 22, 46, 47]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter-to-obtain

$$u=\sum_{i=0}^{\infty} p^i u_i = u_0+pu_1+p^2u_2+\ldots. \tag{10}$$

If $p\to 1$, then (10) corresponds to (7) and becomes the approximate solution of the form

$$f=\lim_{p\to 1} u=\sum_{i=0}^{\infty} u_i. \tag{11}$$

It is well known that the series (10) is convergent for most of the cases and also the rate of convergence is depending on $L(u)$ (see [16 – 18, 21, 45, 46]). The comparisons of like powers of p give solutions of various orders.

4. Variational Homotopy Perturbation Method

To convey the basic idea of the variational homotopy perturbation method [31], we consider the following general nonlinear differential equation $F(u)=g$, where F represents a nonlinear differential operator. The technique consists on decomposing the linear part of F into $L+N$, where L is an operator easily invertible, N is representing the nonlinear term, and g is the forcing term. Thus the equation in the canonical form is

$$Lu+Nu=g. \tag{12}$$

According to the variational iteration method [5, 6, 16, 25, 32], we can construct the correct functional as follows:

$$u_{n+1} = u_n + \int_0^t \lambda(\tau)[Lu_n + N\tilde{u}_n(x,\tau) - g]\mathrm{d}\tau, \quad (13)$$

where λ is a Lagrange multiplier [5], which can be identified optimally via variational iteration theory. The subscripts n denote the n^{th} approximation, $\tilde{u}_n$ is considered as restricted variation, i. e. $\delta\tilde{u} = 0$. We apply the homotopy perturbation method on (13) to get

$$\sum_{n=0}^{\infty} p^n u_n = u_0 + p\int_0^t \lambda(\tau)\left[\sum_{n=0}^{\infty} p^n L(u_n) + \sum_{n=0}^{\infty} p^n N(\tilde{u}_n)\right]\mathrm{d}\tau - \int_0^t \lambda(\tau) g \mathrm{d}\tau. \quad (14)$$

Comparisons of like powers of p give solutions of various orders.

5. Applications

In this section, we solve the (2+1)-dimensional Korteweg-de Vries-Burgers equation (KdVB) (1) and the (1+1)-dimensional Sharma-Tasso-Olver equation (2) by using the variational iteration method (VIM) and the variational homotopy perturbation method (VHPM).

5.1. Solving the (2+1)-Dimensional KdV-Burgers Equation Using VIM

To verify the variational iteration method for (1), let us define $L = ru_{yy}$ and $N = (u_t + uu_x - qu_{xx} + \mu u_{xxx})_x$. Then we can construct the following correct functional:

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^y \lambda(\zeta)\{r(u_n)_{\zeta\zeta} + [(\tilde{u}_n)_t + \tilde{u}_n(\tilde{u}_n)_x - q(\tilde{u}_n)_{xx} + \mu(\tilde{u}_n)_{xxx}]_x\}\mathrm{d}\zeta, \quad (15)$$

where $\lambda(\zeta)$ is a Lagrange multiplier which can be identified optimally via variational iteration method, $\tilde{u}_n$ is considered as restricted variation, i. e. $\delta\tilde{u}_n = 0$,

$$\begin{aligned}\delta u_{n+1} &= \delta u_n + \delta\int_0^y \lambda(\zeta) \cdot\{ru_{\zeta\zeta} + [(\tilde{u}_n)_t + \tilde{u}_n(\tilde{u}_n)_x - q(\tilde{u}_n)_{xx} + \mu(\tilde{u}_n)_{xxx}]_x\}\mathrm{d}\zeta \\ &= \delta u_n + \delta\int_0^y \lambda(\zeta)\{ru_{\zeta\zeta}\}\mathrm{d}\zeta \quad (16) \\ &= \{1 - r\lambda'|_{\zeta=y}\}\delta u_n + r\{\lambda|_{\zeta=y}\}\delta(u_n)_\zeta + \int_0^y r\lambda''(\zeta)\delta u_n \mathrm{d}\zeta.\end{aligned}$$

Consequently, we obtain the following stationary conditions:

$$r(\lambda')_{\zeta=y} = 1, \quad r(\lambda)_{\zeta=y} = 0, \quad r\lambda''(\zeta) = 0. \quad (17)$$

Therefore, the Lagrange multiplier has the form

$$\lambda(\zeta) = \frac{\zeta - y}{r}, \quad r \neq 0. \quad (18)$$

Substituting (18) into the functional equation (15) we obtain

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^y \frac{\zeta - y}{r}\{r(u_n)_{\zeta\zeta} + [(u_n)_t + u_n(u_n)_x - q(u_n)_{xx} + \mu(u_n)_{xxx}]_x\}\mathrm{d}\zeta. \quad (19)$$

For the purpose of illustration of the VIM for solving the KdVB equation (1), we consider the following initial condition [30]:

$$u(x,0,t) = -c + \frac{6q^2}{25\mu} - d^2 r - \frac{3q^2}{25\mu}[1 + \tanh\zeta]^2, \quad (20)$$

where $\zeta = \frac{q(ct+x)}{10\mu}$, the parameters μ, q, r, c, and d are arbitrary constants. To calculate the first term of the VIM series of the KdVB equation (1) we put $n = 0$ in the functional iteration formula (19) and use the initial condition (20), so we get

$$\begin{aligned}u_1(x,y,t) &= u_0(x,y,t) + \int_0^y \frac{\zeta - y}{r}\{r(u_0)_{\zeta\zeta} + [(u_0)_t + u_0(u_0)_x - q(u_0)_{xx} + \mu(u_0)_{xxx}]_x\}\mathrm{d}\zeta \\ &= -c + \frac{6q^2}{25\mu} - d^2 r - \frac{3q^2}{25\mu}[1 + \tanh\zeta]^2 + \frac{3d^2 q^4 y^2 \operatorname{sech}^2\zeta}{2500\mu^3}[2 - 3\operatorname{sech}^2\zeta + 2\tanh\zeta]. \quad (21)\end{aligned}$$

Similarly, we put $n = 1, 2, \ldots$ to obtain $u_2, u_2, \ldots$. Therefore, we have

$$\begin{aligned} u_0 &= -c + \frac{6q^2}{25\mu} - d^2 r - \frac{3q^2}{25\mu}[1+\tanh\zeta]^2, \\ u_1 &= -c + \frac{6q^2}{25\mu} - d^2 r - \frac{3q^2}{25\mu}[1+\tanh\zeta]^2 \\ &\quad + \frac{3d^2q^4y^2\operatorname{sech}^2\zeta}{2500\mu^3}[2-3\operatorname{sech}^2\zeta + 2\tanh\zeta], \\ u_2 &= -c + \frac{6q^2}{25\mu} - d^2 r - \frac{3q^2}{25\mu}[1+\tanh\zeta]^2 \\ &+ \frac{3d^2q^4y^2\operatorname{sech}^2\zeta}{2500\mu^3}[2-3\operatorname{sech}^2\zeta + 2\tanh\zeta] \\ &+ \frac{d^2q^6y^4\operatorname{sech}^8\zeta}{10^8 r\mu^6}\Big[\{\cosh\zeta+\sinh\zeta\}\{-144q^2 \\ &+ 950d^2\mu r\}\cosh\zeta + \{96q^2 + 25\mu r d^2\}\cosh(3\zeta) \\ &- \{912q^2 + 1050\mu r d^2\}\sinh\zeta + \{96q^2 - 825\mu r d^2\} \\ &\cdot \sinh 3\zeta - \mu r d^2\{175\cosh 5\zeta + 225\sinh 5\zeta\}\Big]. \end{aligned} \tag{22}$$

In this manner the other components can be easily obtained. The approximate solution of the KdVB equation (1) takes the following form:

$$\begin{aligned} u(x,y,t) &= -c + \frac{6q^2}{25\mu} - d^2 r - \frac{3q^2}{25\mu}[1+\tanh\zeta]^2 \\ &+ \frac{3d^2q^4y^2\operatorname{sech}^2\zeta}{2500\mu^3}[2-3\operatorname{sech}^2\zeta + 2\tanh\zeta] \\ &+ \frac{d^2q^6y^4\operatorname{sech}^8\zeta}{10^8 r\mu^6}\{\cosh\zeta+\sinh\zeta\}\Big[\{-144q^2 \\ &+ 950d^2\mu r\}\cosh\zeta + \{96q^2 + 25\mu r d^2\}\cosh(3\zeta) \\ &- \{912q^2 + 1050\mu r d^2\}\sinh\zeta + \{96q^2 - 825\mu r d^2\} \\ &\cdot \sinh 3\zeta - \mu r d^2\{175\cosh 5\zeta + 225\sinh 5\zeta\}\Big] + \ldots. \end{aligned} \tag{23}$$

With reference to [30], the exact solution of (1) takes the following form

$$u = -c + \frac{6q^2}{25\mu} - d^2 r - \frac{3q^2}{25\mu}[1+\tanh\gamma]^2, \tag{24}$$

where $\gamma = \frac{q(ct+x+dy)}{10\mu}$.

Furthermore, to examine the accuracy and reliability of the variational iteration method for the KdVB equation (1), we can also consider the following different initial condition [30]:

$$u(x,0,t) = \frac{3q^2}{25\mu}[2+\operatorname{sech}^2\zeta + 2\tanh\zeta], \tag{25}$$

where $\zeta = \left[\frac{3q^3}{125\mu^2} + \frac{5d^2\mu r}{2q}\right]t - \frac{q}{10\mu}x$ and the parameters μ, q, r, and d are arbitrary constants. Again, to find the solution (1) by the VIM we use (19) for $n = 0, 1, 2, \ldots$. Therefore, we have

$$\begin{aligned} u_0 &= \frac{3q^2}{25\mu}[2+\operatorname{sech}^2\zeta + 2\tanh\zeta], \\ u_1 &= \frac{3q^2}{25\mu}[2+\operatorname{sech}^2\zeta + 2\tanh\zeta] \\ &\quad + \frac{3q^2d^2y^2}{100\mu}\operatorname{sech}^2\zeta[2\tanh^2\zeta - 2\tanh\zeta - \operatorname{sech}^2\zeta], \\ u_2 &= \frac{3q^2}{25\mu}[2+\operatorname{sech}^2\zeta + 2\tanh\zeta] \\ &\quad + \frac{3q^2d^2y^2}{100\mu}\operatorname{sech}^2\zeta[2\tanh^2\zeta - 2\tanh\zeta - \operatorname{sech}^2\zeta] \\ &\quad + \frac{d^2q^2y^4 10^{-6}}{4r\mu^4}\operatorname{sech}^8\zeta[\sinh\zeta - \cosh\zeta] \\ &\quad \cdot \Big\{d^2 r\mu^3(4375\cosh(5\zeta) - 5625\sinh(5\zeta)) \\ &\qquad - (625d^2r\mu^3 + 96\mu^4)\cosh(3\zeta) \\ &\qquad + (144\mu^4 - 23750d^2r\mu^3)\cosh\zeta \\ &\qquad + (96\mu^4 - 20625d^2r\mu^3)\sinh(3\zeta) \\ &\qquad - (912\mu^4 - 26250d^2r\mu^3)\sinh\zeta\Big\}, \end{aligned} \tag{26}$$

and so on. Then the approximate solution has the form

$$\begin{aligned} u(x,y,t) &= \frac{3q^2}{25\mu}[2+\operatorname{sech}^2\zeta + 2\tanh\zeta] \\ &+ \frac{3q^2d^2y^2}{100\mu}\operatorname{sech}^2\zeta[2\tanh^2\zeta - 2\tanh\zeta - \operatorname{sech}^2\zeta] \\ &+ \frac{d^2q^2y^4 10^{-6}}{4r\mu^4}\operatorname{sech}^8\zeta[\sinh\zeta - \cosh\zeta] \\ &\cdot \Big\{d^2 r\mu^3(4375\cosh(5\zeta) - 5625\sinh(5\zeta)) \\ &\quad - (625d^2r\mu^3 + 96\mu^4)\cosh(3\zeta) \\ &\quad + (144\mu^4 - 23750d^2r\mu^3)\cosh\zeta \\ &\quad + (96\mu^4 - 20625d^2r\mu^3)\sinh(3\zeta) \\ &\quad - (912\mu^4 - 26250d^2r\mu^3)\sinh\zeta\Big\} + \ldots. \end{aligned} \tag{27}$$

With reference to [30], the exact solution of (1) under the initial condition (25) takes the following form:

$$u(x,y,t) = \frac{3q^2}{25\mu}[2+\operatorname{sech}^2\gamma + 2\tanh\gamma], \tag{28}$$

where $\gamma = \left[\frac{3q^3}{125\mu^2} + \frac{5d^2\mu r}{2q}\right]t - \frac{q}{10\mu}x + \frac{d}{2}y$.

5.2. Solving the (1+1)-Dimensional Sharma-Tasso-Olver Equation Using the VIM

To verify the variational iteration method for (2), we construct the following correct functional:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\Big[(u_n)_\tau + \alpha(u_n^3)_x + \frac{3\alpha}{2}(u_n^2)_{xx} + \alpha(u_n)_{xxx}\Big]\mathrm{d}\tau. \tag{29}$$

Making the correct functional stationary, the Lagrange multipliers can be identified as $\lambda(\tau) = -1$, the correct functional of the Sharma-Tasso-Olver equation is

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \Big[(u_n)_\tau + \alpha(u_n^3)_x + \frac{3\alpha}{2}(u_n^2)_{xx} + \alpha(u_n)_{xxx}\Big]\mathrm{d}\tau. \tag{30}$$

For the purpose of illustration of the VIM for solving the Sharma-Tasso-Olver equation (2), we consider the following initial condition [10]:

$$u(x,0) = \frac{1}{2}\left[1 + \mathrm{i}\frac{\sin(kx)}{1+\cos(kx)}\right], \tag{31}$$

where k is an arbitrary constant. To calculate the terms of the VIM series of the Sharma-Tasso-Olver equation (2), we put $n = 0,1,2,\ldots$ in the functional iteration formula (30) and use the initial condition (31). Consequently, we have

$$\begin{aligned} u_0(x,t) &= \frac{1}{2}\left[1 + \mathrm{i}\frac{\sin(kx)}{1+\cos(kx)}\right], \\ u_1(x,t) &= \frac{1}{2}\left[1 + \mathrm{i}\frac{\sin(kx)}{1+\cos(kx)}\right] \\ &+ \mathrm{i}\alpha t k^2\left\{\frac{3\sin(kx)}{4(1+\cos(kx))} - \frac{9\sin(kx)\cos(kx)}{4(1+\cos(kx))^2} - \frac{3\sin^3(kx)}{2(1+\cos(kx))^3}\right\}, \end{aligned} \tag{32}$$

and so on. Then, the approximate solution takes the form

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[1 + \mathrm{i}\frac{\sin(kx)}{1+\cos(kx)}\right] \\ &+ \mathrm{i}\alpha t k^2\left\{\frac{3\sin(kx)}{4(1+\cos(kx))} - \frac{9\sin(kx)\cos(kx)}{4(1+\cos(kx))^2} - \frac{3\sin^3(kx)}{2(1+\cos(kx))^3}\right\} + \ldots. \end{aligned} \tag{33}$$

With reference to [10], the exact solution of (2) takes the following form:

$$u(x,t) = \frac{1}{2}\left[1 + \mathrm{i}\frac{\sin(kx - \alpha k^2 t)}{1+\cos(kx - \alpha k^2 t)}\right], \tag{34}$$

where k is an arbitrary constant.

5.3. Solving the (2+1)-Dimensional KdV-Burgers Equation Using VHPM

The main purpose of the work reported in this subsection is testing of the VHPM based on the method introduced in Section 4. To convey the basic idea of the variational homotopy perturbation method for (1), let us consider the functional iteration formula

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^y \frac{\zeta - y}{r}\big\{r(u_n)_{\zeta\zeta} + [(u_n)_t + u_n(u_n)_n - q(u_n)_{xx} + \mu(u_n)_{xxx}]_x\big\}\mathrm{d}\zeta. \tag{35}$$

We apply the homotopy perturbation method for the previous functional iteration formula to get

$$\begin{aligned} \sum_{n=0}^\infty p^n u_n &= u_0(x,y,t) + p\int_0^y \frac{\zeta - y}{r}\Bigg\{r\left(\sum_{n=0}^\infty p^n u_n\right)_{\zeta\zeta} \\ &+ \Bigg[\left(\sum_{n=0}^\infty p^n u_n\right)_t + \sum_{n=0}^\infty p^n u_n\left(\sum_{n=0}^\infty p^n u_n\right)_x \\ &- q\left(\sum_{n=0}^\infty p^n u_n\right)_{xx} + \mu\left(\sum_{n=0}^\infty p^n u_n\right)_{xxx}\Bigg]_x\Bigg\}\mathrm{d}\zeta. \end{aligned} \tag{36}$$

Comparing the coefficients of like powers of p^0, p^1, p^2, p^3, ..., and use the initial condition (20) we have

$$\begin{aligned} u_0(x,y,t) &= u(x,0,t), \\ u_1 &= \int_0^y \frac{\zeta - y}{r}\big\{[(u_0)_t + u_0(u_0)_x - q(u_0)_{xx} + \mu(u_0)_{xxx}]_x\big\}\mathrm{d}\zeta, \end{aligned}$$

$$u_2 = \int_0^y \frac{\zeta - y}{r}\{r(u_1)_{\zeta\zeta} + [(u_1)_t + (u_0(u_1)_x + u_1(u_0)_x) - q(u_1)_{xx} + \mu(u_1)_{xxx}]_x\}\mathrm{d}\zeta,$$

$$u_3 = \int_0^y \frac{\zeta - y}{r}\{r(u_2)_{\zeta\zeta} + [(u_2)_t + (u_0(u_2)_x + u_2(u_0)_x + u_1(u_1)_x) - q(u_2)_{xx} + \mu(u_2)_{xxx}]_x\}\mathrm{d}\zeta. \quad (37)$$

The other components can be found similarly. After some reduction, we have

$$u_0 = -c\frac{6q^2}{25\mu} - d^2 r - \frac{3q^2}{25\mu}[1+\tanh\zeta]^2,$$

$$u_1 = +\frac{3d^2q^4y^2\operatorname{sech}^2\zeta}{2500\mu^3}[2 - 3\operatorname{sech}^2\zeta + 2\tanh\zeta],$$

$$u_2 = \frac{d^2q^6y^4\operatorname{sech}^8\zeta}{10^8 r\mu^6}\{\cosh\zeta + \sinh\zeta\}\Big[\{-144q^2 + 950d^2\mu r\}\cosh\zeta + \{96q^2 + 25\mu r d^2\}\cosh(3\zeta) - \{912q^2 + 1050\mu r d^2\}\sinh\zeta + \{96q^2 - 825\mu r d^2\} \cdot \sinh 3\zeta - \mu r d^2\{175\cosh 5\zeta + 225\sinh 5\zeta\}\Big]. \quad (38)$$

Then the approximate solution of (1) under the initial condition (20) by the VHPM has the form

$$u(x,y,t) = -c\frac{6q^2}{25\mu} - d^2 r - \frac{3q^2}{25\mu}[1+\tanh\zeta]^2 + \frac{3d^2q^4y^2\operatorname{sech}^2\zeta}{2500\mu^3}[2 - 3\operatorname{sech}^2\zeta + 2\tanh\zeta] + \frac{d^2q^6y^4\operatorname{sech}^8\zeta}{10^8 r\mu^6}\{\cosh\zeta + \sinh\zeta\}\Big[\{-144q^2 + 950d^2\mu r\}\cosh\zeta + \{96q^2 + 25\mu r d^2\}\cosh(3\zeta) - \{912q^2 + 1050\mu r d^2\}\sinh\zeta + \{96q^2 - 825\mu r d^2\} \cdot \sinh 3\zeta - \mu r d^2\{175\cosh 5\zeta + 225\sinh 5\zeta\}\Big] + \dots \quad (39)$$

To demonstrate the convergence of the variational iteration method and the variational homotopy perturbation method, the result of the numerical example is presented and only few terms are required to obtain the accurate solution. The accuracy of the approximate solution for (1) under the initial condition (20) is controlable, and the absolute errors are very small with the present choice of x, y, t. These results are listed in Table 1. The numerical results are much closer to the corresponding exact solution with the initial condition. Both the exact results and the approximate solutions of (1) under the initial condition (20) are plotted in Figures 1 and 2. The numerical results are much closer to

Table 1. Approximate solution $u(x,y,t)$ for (1) with the initial condition (20) in comparison with the exact solution when $c = \mu = q = 1$, $r = 1.5$, $d = 0.001$, $t = 10$, and $y = 10$.

x	u_{exact}	u_{app}	$\lvert u_{\text{exact}} - u_{\text{app}}\rvert$
−50	−0.760002	−0.760002	2.158507E-10
−40	−0.760004	−0.760004	1.7096273E-8
−30	−0.760157	−0.760157	6.09957614E-7
−20	−0.766846	−0.766822	2.40305310E-5
−10	−0.880242	−0.880002	2.39996719E-4
00	−1.13256	−1.13239	1.77561430E-4
10	−1.22292	−1.22289	3.33028872E-5
20	−1.23764	−1.23763	4.72400106E-6
30	−1.23968	−1.23968	6.43441089E-7
40	−1.23996	−1.23996	8.7156055E-8
50	−1.24	−1.24	1.17966783E-8

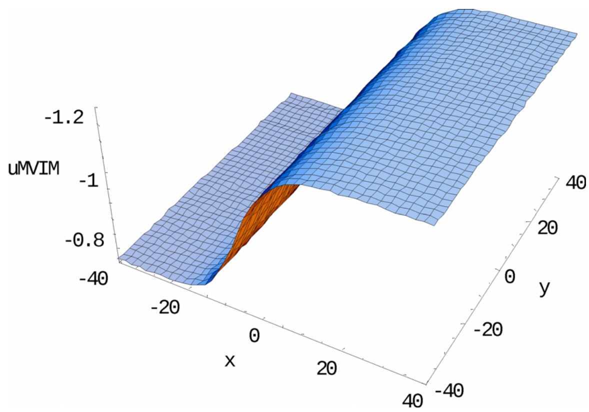

Fig. 1. Approximate solution for the (2+1)-dimensional Korteweg-de Vries-Burgers equation (1) under the initial condition (20) at $c = \mu = q = r = 1$, $d = 0.001$, and $t = 10$.

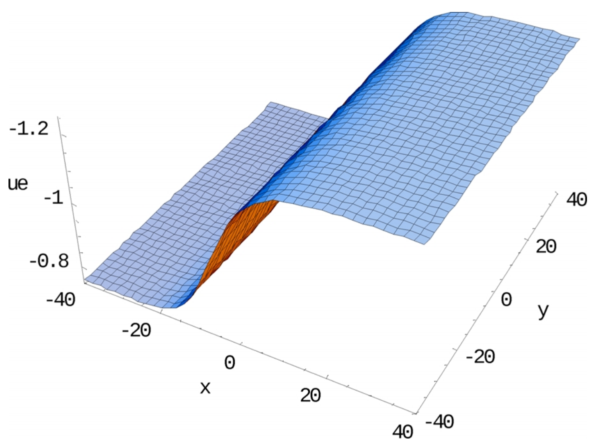

Fig. 2. Exact solution for the (2+1)-dimensional Korteweg-de Vries-Burgers equation (1) under the initial condition (20) at $c = \mu = q = r = 1$, $d = 0.001$, and $t = 10$.

the corresponding exact solution with the initial condition.

For the purpose of illustration of the variational homotopy perturbation method for solving the KdVB equation (1) subject to the initial condition (25), we use the components (37) with the initial condition (25). After same reduction, we have

$$\begin{aligned} u_0 &= \frac{3q^2}{25\mu}[2+\operatorname{sech}^2\zeta+2\tanh\zeta], \\ u_1 &= \frac{3q^2d^2y^2}{100\mu}\operatorname{sech}^2\zeta[2\tanh^2\zeta-2\tanh\zeta-\operatorname{sech}^2\zeta], \\ u_2 &= \frac{d^2q^2y^4 10^{-6}}{4r\mu^4}\operatorname{sech}^8\zeta[\sinh\zeta-\cosh\zeta] \\ &\cdot\Big\{d^2r\mu^3(4375\cosh(5\zeta)-5625\sinh(5\zeta)) \\ &\quad-(625d^2r\mu^3+96\mu^4)\cosh(3\zeta) \\ &\quad+(144\mu^4-23750d^2r\mu^3)\cosh\zeta \\ &\quad+(96\mu^4-20625d^2r\mu^3)\sinh(3\zeta) \\ &\quad-(912\mu^4-26250d^2r\mu^3)\sinh\zeta\Big\}, \end{aligned} \tag{40}$$

and so on. Then the approximate solution of (1) under the initial condition (25) by the VHPM has the form

$$\begin{aligned} u(x,y,t) &= \frac{3q^2}{25\mu}[2+\operatorname{sech}^2\zeta+2\tanh\zeta] \\ &+\frac{3q^2d^2y^2}{100\mu}\operatorname{sech}^2\zeta[2\tanh^2\zeta-2\tanh\zeta-\operatorname{sech}^2\zeta] \\ &+\frac{d^2q^2y^4 10^{-6}}{4r\mu^4}\operatorname{sech}^8\zeta[\sinh\zeta-\cosh\zeta] \\ &\cdot\Big\{d^2r\mu^3(4375\cosh(5\zeta)-5625\sinh(5\zeta)) \\ &\quad-(625d^2r\mu^3+96\mu^4)\cosh(3\zeta) \\ &\quad+(144\mu^4-23750d^2r\mu^3)\cosh\zeta \\ &\quad+(96\mu^4-20625d^2r\mu^3)\sinh(3\zeta) \\ &\quad-(912\mu^4-26250d^2r\mu^3)\sinh\zeta\Big\}+\ldots, \end{aligned} \tag{41}$$

where $\zeta=\left[\frac{3q^3}{125\mu^2}+\frac{5d^2r\mu}{2q}\right]t-\frac{qx}{10\mu}$.

Both the exact results and the approximate solutions of (1) under the initial condition (25) are plotted in Figures 3 and 4. The numerical results are much closer to the corresponding exact solution with the initial condition.

The accuracy of the approximate solution for (1) under the initial condition (25) is controlable, and the absolute errors are very small with the present choice of x, y, t. These results are listed in Table 2.

Table 2. Approximate solution $u(x,y,t)$ for (1) with the initial condition (25) in comparison with the exact solution when $\mu=q=1$, $r=1.5$, $d=0.001$, $t=y=10$.

x	u_{exact}	u_{app}	$\lvert u_{\text{exact}}-u_{\text{app}}\rvert$
−50	0.0000711402	0.0000704377	4.5764366E-7
−40	0.48	0.48	4.9707700E-9
−30	0.479999	0.479999	3.6224875E-8
−20	0.479941	0.479940	2.4122393E-7
−10	0.477187	0.477135	7.0268036E-5
00	0.410733	0.409867	4.2488720E-4
10	0.157995	0.156837	2.9695880E-3
20	0.0274654	0.0272056	1.5244874E-4
30	0.0038611	0.0038232	2.4359285E-5
40	0.000525286	0.000520102	3.3713263E-6
50	0.0000711402	0.0000704377	4.5764366E-7

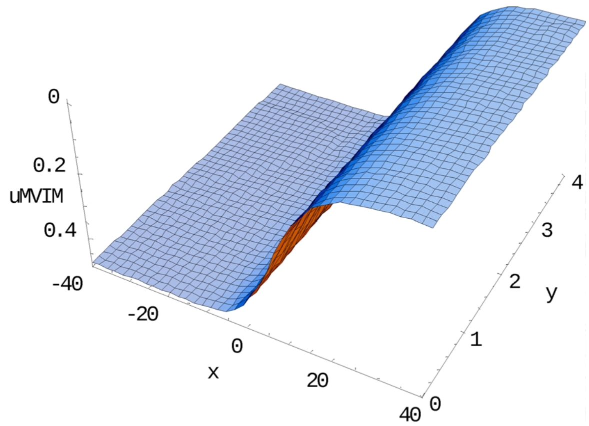

Fig. 3. Approximate solution for the (2+1)-dimensional Korteweg-de Vries-Burgers equation (1) under the initial condition (25) at $\mu=q=r=1$, $d=0.001$, and $t=10$.

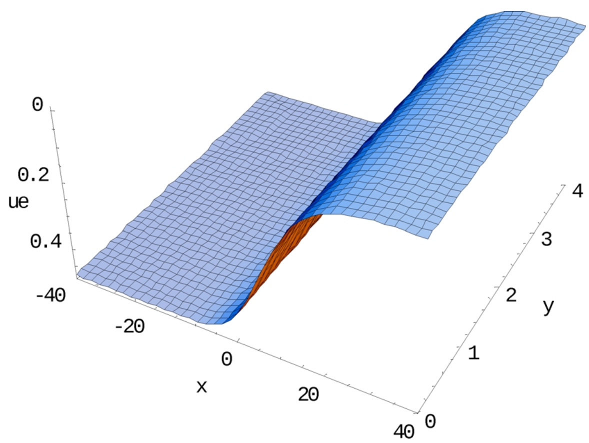

Fig. 4. Exact solution for the (2+1)-dimensional Korteweg-de Vries-Burgers equation (1) under the initial condition (25) at $\mu=q=r=1$, $d=0.001$, and $t=10$.

5.4. Solving the (1+1)-Dimensional Sharma-Tasso-Olver Using VHPM

By the same way of the previous subsection we apply the VHPM for (2) to get

$$\sum_{n=0}^{\infty} p^n u_n = u_0 - p\int_0^t \Bigg[\Big(\sum_{n=0}^{\infty} p^n u_n\Big)_\tau + \alpha\Big(\Big[\sum_{n}^{\infty} p^n u_n\Big]^3\Big)_x + \frac{3\alpha}{2}\Big(\Big[\sum_{n=0}^{\infty} p^n u_n\Big]^2\Big)_{xx} + \alpha\Big(\sum_{n}^{\infty} p^n u_n\Big)_{xxx}\Bigg]\mathrm{d}\tau, \tag{42}$$

Comparing the coefficients of like parameters p^0, p^1, p^2, ..., then we have

$$\begin{aligned} u_0(x,t) &= u(x,0),\\ u_1(x,t) &= -\int_0^t [\alpha(u_0^3)_x + \frac{3\alpha}{2}(u_0^2)_{xx} + \alpha(u_0)_{xxx}]\mathrm{d}\tau,\\ u_2(x,t) &= -\int_0^t [(u_1)_\tau + \alpha(3u_0^2u_1)_x + \frac{3\alpha}{2}(2u_0u_1)_{xx} + \alpha(u_1)_{xxx}]\mathrm{d}\tau. \end{aligned} \tag{43}$$

The other component can be found similarly. After some reduction, we have

$$\begin{aligned} u_0(x,t) &= \frac{1}{2}\left[1 + \mathrm{i}\frac{\sin(kx)}{1+\cos(kx)}\right],\\ u_1(x,t) &= \mathrm{i}\alpha t k^2\left\{\frac{3\sin(kx)}{4(1+\cos(kx))} - \frac{9\sin(kx)\cos(kx)}{4(1+\cos(kx))^2} - \frac{3\sin^3(kx)}{2(1+\cos(kx))^3}\right\}, \end{aligned} \tag{44}$$

and so on. Then the VHPM solution of (2) takes the form

$$u(x,t) = \frac{1}{2}\left[1 + \mathrm{i}\frac{\sin(kx)}{1+\cos(kx)}\right] + \mathrm{i}\alpha t k^2\left\{\frac{3\sin(kx)}{4(1+\cos(kx))} - \frac{9\sin(kx)\cos(kx)}{4(1+\cos(kx))^2} - \frac{3\sin^3(kx)}{2(1+\cos(kx))^3}\right\} + \dots. \tag{45}$$

To demonstrate the convergence of the variational iteration method and the variational homotopy perturbation method, the result of the numerical example is presented and only few terms are required to obtain the accurate solution. The accuracy of the approximate solution for (2) under the initial condition (31) is controlable, and the absolute errors are very small with the

Table 3. Numerical absolute solution $u(x,t)$ for (2) with the initial condition (31) in comparison with the exact solution when $\alpha = k = 0.01$, and $t = 2$.

x	u_{exact}	u_{app}	$\lvert u_{\text{exact}} - u_{\text{app}}\rvert$
−50	0.500156	0.500156	5.74260777E-7
−40	0.51017	0.510162	7.77789844E-6
−30	0.505678	0.505673	5.54948352E-6
−20	0.502511	0.502507	3.48755898E-6
−10	0.500626	0.500624	1.53072305E-6
00	0.5	0.5	3.76406186E-7
10	0.500626	0.500628	2.28637202E-6
20	0.50251	0.502515	4.25179659E-6
30	0.505678	0.505684	6.32830261E-6
40	0.510169	0.510178	8.57770837E-6
50	0.516042	0.516053	1.10718013E-5

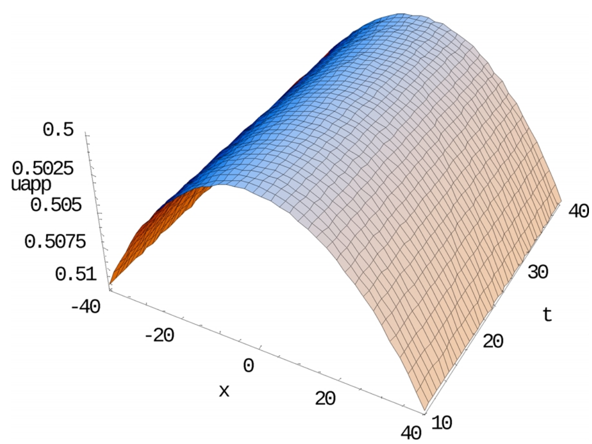

Fig. 5. Absolute of the numerical solution for the (1+1)-dimensional Sharma-Tasso-Olver equation (2) under the initial condition (31) at $\alpha = k = 0.01$.

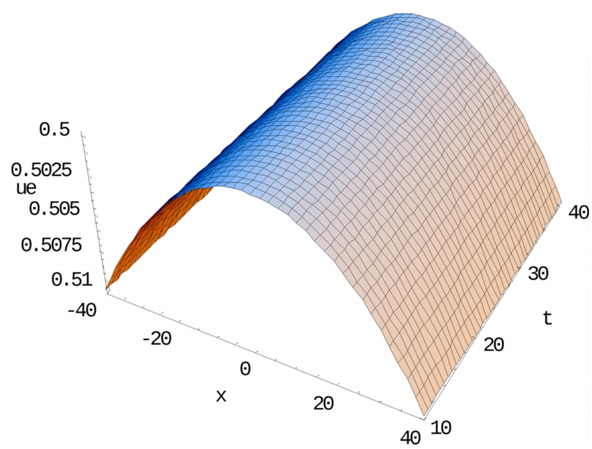

Fig. 6. Absolute of the exact solution for the (1+1)-dimensional Sharma-Tasso-Olver equation (2) under the initial condition (31) at $\alpha = k = 0.01$.

present choice of x, t. These results are listed in Table 3.

Both the exact results and the approximate solutions of (2) under the initial condition (31) are plotted in Figures 5 and 6. The numerical results are much closer to the corresponding exact solution with the initial condition.

6. Conclusions

In the present article, the variational iteration method and the variational homotopy perturbation method are used for finding the solution of the (2+1)-dimensional KdV-Burgers equation with two different initial conditions and the (1+1)-dimensional Sharma-Tasso-Olver equation with one initial condition. These methods are very powerful and efficient techniques in finding the approximate solutions for wide classes of nonlinear problems. These methods also present a rapid convergence solution in comparing with other methods.

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